

A measure of multipartite entanglement with computable lower bounds

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In this paper, we introduce a measure of multipartite entanglement, k -ME concurrence $C_{k-\text{ME}}(\rho)$, which satisfies important characteristics of an entanglement measure. Two lower bounds on this measure are given. These lower bounds are experimentally implementable without quantum state tomography and are easily computable as no optimization or eigenvalue evaluation is needed. We illustrate detailed examples in which the given bounds perform better than other known detect criteria.

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I. INTRODUCTION

Entanglement as a physical resource plays an important role in quantum information, such as, quantum communication [1–7] and quantum computing [8, 9]. So it is a significant work to quantify entanglement not only in theoretical research but also in practical application. But the researchers encounter with tremendous challenges in introducing entanglement measure to quantify entanglement. The concurrence is an accepted entanglement measure for bipartite qubit states [10], and is also defined for bipartite high dimensional states [11], but it is not computable because of optimization for bipartite high dimensional mixed states. For multipartite quantum systems, although there are some criteria [12–20] to detect genuine multipartite entanglement, but there is not computable measure quantifying the amount of genuine multipartite entanglement in general. Zhi-Hao Ma [21] *et al.* defined a generalized concurrence called gme-concurrence which satisfies the necessary conditions for genuine multipartite entanglement measure [23, 24]. Although for general mixed states it is not computable owing to the optimization, they gave a lower bound [21, 22].

In this paper, we define a generalized concurrence (k -ME concurrence) for a finite-dimensional systems of arbitrarily many parties as an entanglement measure. We show that strong lower bounds on this measure can be derived by exploiting close analytic relations between this concurrence and recently introduced detection criteria for genuine multipartite entanglement [16–18]. And then we provide examples in which the entanglement criteria based on our lower bound have better performance with respect to the known methods. the lower bound obtained by Ref. [21].

II. MULTIPARTITE ENTANGLEMENT

Before we state our lower bounds, an introduction of concepts and notations that will be involved in the subsequent sections of our article is necessary. Throughout the paper, we consider a multiparticle quantum system $\mathcal{H} = \otimes_{i=1}^n \mathcal{H}_i = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ with n parts of respective dimension d_i , $i = 1, 2, \cdots, n$. A k -partition $A_1|A_2|\cdots|A_k$ (of $\{1, 2, \cdots, n\}$) means that the set $\{A_1, A_2, \cdots, A_k\}$ is a collection of pairwise disjoint sets, and the union of all sets in $\{A_1, A_2, \cdots, A_k\}$ is $\{1, 2, \cdots, n\}$ (disjoint union $\bigcup_{i=1}^k A_i = \{1, 2, \cdots, n\}$). An pure state $|\psi\rangle$ of a n -partite quantum system \mathcal{H} is called k -separable if there is a k -partition $A_1|A_2|\cdots|A_k = j_1^1 \cdots j_{n_1}^1 | j_1^2 \cdots j_{n_2}^2 | \cdots | j_1^k \cdots j_{n_k}^k$ such that

$$|\psi\rangle = |\psi_1\rangle_{A_1} |\psi_2\rangle_{A_2} \cdots |\psi_k\rangle_{A_k}, \quad (1)$$

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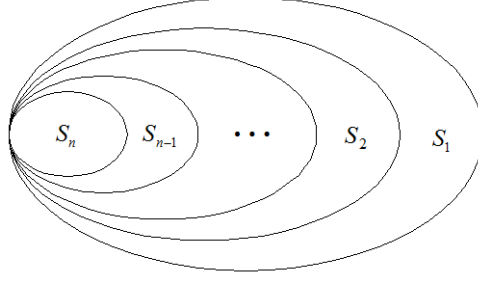


FIG. 1: (Color online). Illustration of the convex nested structure of the sets S_k of all k -separable states. Each set is convexly embedded within the next set: $S_n \subset S_{n-1} \subset \dots \subset S_2 \subset S_1$, and the complement $S_1 \setminus S_k$ of S_k in S_1 is the set of all k -nonseparable states.

where $|\psi_i\rangle_{A_i}$ is the state of subsystem A_i , and disjoint union $\bigcup_{t=1}^k A_t = \bigcup_{t=1}^k \{j_1^t, j_2^t, \dots, j_{n_t}^t\} = \{1, 2, \dots, n\}$. An n -partite mixed state ρ is k -separable if it can be written as a convex combination of k -separable pure states

$$\rho = \sum_m p_m |\psi_m\rangle\langle\psi_m|, \quad (2)$$

where $\{|\psi_m\rangle\}$ might be k -separable with respect to different partitions. Thus, a mixed k -separable state does not need to be separable under any particular k -partition of the Hilbert space. In general, k -separable mixed states are not separable with regard to any specific partition. If an n -partite state is not 2-separable (biseparable), then it is called genuinely n -partite entangled. It is called fully separable, iff it is n -separable.

Note that whenever a state is k -separable, it is automatically also k' -separable for all $k' < k$. If we denote the set of all k -separable states by S_k ($k = 2, 3, \dots, n$) and the set of all states by S_1 , then each set S_k is convex and embedded within the next set: $S_n \subset S_{n-1} \subset \dots \subset S_2 \subset S_1$, and the complement $S_1 \setminus S_k$ of S_k in S_1 is the set of all k -nonseparable states. In particular, the complement $S_1 \setminus S_2$ is the set of all genuine n -partite entangled (2-nonseparable) states. We can illustrate the convex nested structure of multipartite entanglement (see Fig. 1).

III. A MEASURE OF MULTIPARTITE ENTANGLEMENT AND ITS LOWER BOUNDS

Let us now introduce a measure of multipartite entanglement (k -nonseparable). For n -partite pure state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_n$, where $\dim \mathcal{H}_l = d_l$, $l = 1, 2, \dots, n$, we define the k -ME-concurrence as

$$C_{k\text{-ME}}(|\psi\rangle) = \min_A \sqrt{2 \left(1 - \frac{\sum_{t=1}^k \text{Tr}(\rho_{A_t}^2)}{k} \right)} = \min_A \sqrt{\frac{2 \sum_{t=1}^k (1 - \text{Tr}(\rho_{A_t}^2))}{k}}, \quad (3)$$

where $\rho_{A_t} = \text{Tr}_{\bar{A}_t}(|\psi\rangle\langle\psi|)$ is the reduce density matrix of subsystem A_t (\bar{A}_t is the complement of A_t in $\{1, 2, \dots, n\}$), and the minimum is taken over all possible k -partitions $A = A_1 | \dots | A_k$ of $\{1, 2, \dots, n\}$.

For n -partite mixed state ρ , we define the k -ME-concurrence as

$$C_{k\text{-ME}}(\rho) = \inf_{\{p_m, |\psi_m\rangle\}} \sum_m p_m C_{k\text{-ME}}(|\psi_m\rangle). \quad (4)$$

where the infimum is taken over all possible pure states decompositions $\rho = \sum_m p_m |\psi_m\rangle\langle\psi_m|$. Note that the GME concurrence [21] is our special case $C_{2\text{-ME}}(\rho)$.

k -ME-concurrence $C_{k\text{-ME}}(\rho)$, a measure of multipartite entanglement, satisfies the following useful properties:

- M1 $C_{k\text{-ME}}(\rho) = 0$ for any $\rho \in S_k$ (zero for all k -separable states).
- M2 $C_{k\text{-ME}}(\rho) > 0$ for any $\rho \in S_1 \setminus S_k$ (strictly greater than zero for all k -nonseparable states).
- M3 $C_{k\text{-ME}}(\sum_i p_i \rho_i) \leq \sum_i p_i C_{k\text{-ME}}(\rho_i)$ (convexity)
- M4 $C_{k\text{-ME}}(\Lambda_{\text{LOCC}}(\rho)) \leq C_{k\text{-ME}}(\rho)$ (nonincreasing under local operations and classical communication (LOCC)).
- M5 $C_{k\text{-ME}}(U_{\text{Local}}^\dagger \rho U_{\text{Local}}) = C_{k\text{-ME}}(\rho)$ (invariant under local unitary transformations).
- M6 $C_{k\text{-ME}}(\rho \otimes \sigma) \leq C_{k\text{-ME}}(\rho) + C_{k\text{-ME}}(\sigma)$ (subadditivity).

IV. LOWER BOUNDS

A. Statement of results

Let $|\phi(x)\rangle = \otimes_{i=1}^n |x_i\rangle = |x_1 x_2 \cdots x_n\rangle$ be a fully separable state on Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \mathcal{H}_n$, and $|\Phi_{ij}(x)\rangle = |\phi_i(x)\rangle |\phi_j(x)\rangle$ a product state in $\mathcal{H}^{\otimes 2} = (\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \mathcal{H}_n)^{\otimes 2}$, where $|\phi_i(x)\rangle = |x_1 x_2 \cdots x_{i-1} x'_i x_{i+1} \cdots x_n\rangle$ and $|\phi_j(x)\rangle = |x_1 x_2 \cdots x_{j-1} x'_j x_{j+1} \cdots x_n\rangle$ be the product states obtained from $|\phi(x)\rangle$ by applying (independently) local unitary transformations to $|x_i\rangle \in \mathcal{H}_i$ and $|x_j\rangle \in \mathcal{H}_j$, respectively. Let P denotes the operator that performs a simultaneous local permutation on all subsystems in $(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n)^{\otimes 2}$, while P_i just performs a permutation on $\mathcal{H}_i^{\otimes 2}$ and leaves all other subsystems unchanged. Let

$$I_k(\rho, \phi(x)) = \sum_{i \neq j} \sqrt{\langle \Phi_{ij}(x) | \rho^{\otimes 2} P | \Phi_{ij}(x) \rangle} - \sum_{i \neq j} \sqrt{\langle \Phi_{ij}(x) | P_i^+ \rho^{\otimes 2} P_i | \Phi_{ij}(x) \rangle} - (n-k) \sum_i \sqrt{\langle \Phi_{ii}(x) | P_i^+ \rho^{\otimes 2} P_i | \Phi_{ii}(x) \rangle}, \quad (5)$$

then we have the following bounds.

Bound 1.

$$C_{k-\text{ME}}(\rho) \geq H_k I_k(\rho, \phi(x)), \quad (6)$$

where

$$H_k = \min_A \frac{\sqrt{k}}{\sqrt{\sum_{t=1}^k n_t(n - n_t)}}. \quad (7)$$

Here the minimum is taken over all possible k -partitions $A = A_1 | \cdots | A_k$ of $\{1, 2, \cdots, n\}$.

Bound 2.

$$C_{k-\text{ME}}(\rho) \geq \max_{\{\phi(x) \neq \phi(y)\}} \bar{H}_k(I(\rho, \phi(x)) + I(\rho, \phi(y))). \quad (8)$$

where

$$\bar{H}_k = \min_{\gamma} \frac{\sqrt{k}}{\sqrt{2 \sum_{t=1}^k n_t(n - n_t)}} = \frac{1}{\sqrt{2}} H_k. \quad (9)$$

B. Proof

Any pure quantum state of an n particle system can be denoted by vectors in Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$, as follows:

$$|\psi\rangle = \sum_{i_1, \dots, i_n} c_{i_1, \dots, i_n} |i_1 \cdots i_n\rangle, \quad (10)$$

which can be rewritten as

$$|\psi\rangle = \sum_{\gamma_{A_t}, \gamma_{\bar{A}_t}} c_{\gamma_{A_t} \gamma_{\bar{A}_t}} |\gamma_{A_t} \gamma_{\bar{A}_t}\rangle, \quad (11)$$

where $\{|i_j\rangle\}$ is the orthonormal basis of \mathcal{H}_k , and a basis vector of subsystem A_t is denoted by $|\gamma_{A_t}\rangle = |i_{j_1^t} \cdots i_{j_{n_t}^t}\rangle$. Here $A_1 | A_2 | \cdots | A_k = j_1^1 \cdots j_{n_1}^1 | j_1^2 \cdots j_{n_2}^2 | \cdots | j_1^k \cdots j_{n_k}^k$ is a k -partition of $\{1, 2, \cdots, n\}$, and \bar{A}_t is the complement of subsystem A_t in $\{1, 2, \cdots, n\}$. Thus,

$$\rho_{A_t} = \text{Tr}_{\bar{A}_t}(|\psi\rangle\langle\psi|) = \sum_{\gamma_{A_t}, \eta_{A_t}} \left(\sum_{\gamma_{\bar{A}_t}} c_{\gamma_{A_t} \gamma_{\bar{A}_t}} c_{\eta_{A_t} \gamma_{\bar{A}_t}}^* \right) |\gamma_{A_t}\rangle\langle\eta_{A_t}| = \sum_{\gamma_{A_t}, \eta_{A_t}} \rho_{\gamma_{A_t}, \eta_{A_t}} |\gamma_{A_t}\rangle\langle\eta_{A_t}|, \quad (12)$$

and

$$\begin{aligned}\text{Tr}(\rho_{A_t}^2) &= \sum_{\gamma_{A_t}, \eta_{A_t}} |\rho_{\gamma_{A_t}, \eta_{A_t}}|^2 \\ &= \sum_{\gamma_{A_t}} |\rho_{\gamma_{A_t}, \gamma_{A_t}}|^2 + 2 \sum_{s_{\gamma_{A_t}} < s_{\eta_{A_t}}} |\rho_{\gamma_{A_t}, \eta_{A_t}}|^2,\end{aligned}\quad (13)$$

where $s_{\gamma_{A_t}} = \sum_{l=1}^{n_t} i_{j_l^t} d_{j_l^t+1} d_{j_l^t+2} \cdots d_n d_{n+1}$ and $d_{n+1} = 1$. It follows that

$$\begin{aligned}1 - \text{Tr}(\rho_{A_t}^2) &= \sum_{\gamma_{A_t}} \rho_{\gamma_{A_t}, \gamma_{A_t}} (1 - \rho_{\gamma_{A_t}, \gamma_{A_t}}) - 2 \sum_{s_{\gamma_{A_t}} < s_{\eta_{A_t}}} |\rho_{\gamma_{A_t}, \eta_{A_t}}|^2 \\ &= 2 \sum_{s_{\gamma_{A_t}} < s_{\eta_{A_t}}} (\rho_{\gamma_{A_t}, \gamma_{A_t}} \rho_{\eta_{A_t}, \eta_{A_t}} - |\rho_{\gamma_{A_t}, \eta_{A_t}}|^2) \\ &= 2 \sum_{s_{\gamma_{A_t}} < s_{\eta_{A_t}}} \left(\sum_{\gamma_{A_t}, \eta_{A_t}} |c_{\gamma_{A_t} \gamma_{A_t}} c_{\eta_{A_t} \eta_{A_t}}|^2 - \sum_{\gamma_{A_t}, \eta_{A_t}} c_{\gamma_{A_t} \gamma_{A_t}} c_{\eta_{A_t} \eta_{A_t}} c_{\eta_{A_t} \gamma_{A_t}}^* c_{\gamma_{A_t} \eta_{A_t}}^* \right) \\ &= 2 \sum_{s_{\gamma_{A_t}} < s_{\eta_{A_t}}} \sum_{s_{\gamma_{A_t}} < s_{\eta_{A_t}}} |c_{\gamma_{A_t} \gamma_{A_t}} c_{\eta_{A_t} \eta_{A_t}} - c_{\eta_{A_t} \gamma_{A_t}} c_{\gamma_{A_t} \eta_{A_t}}|^2.\end{aligned}\quad (14)$$

1. Bound 1

From (14) we have

$$\begin{aligned}\frac{2 \sum_{t=1}^k (1 - \text{Tr}(\rho_{A_t}^2))}{k} &= \frac{4 \sum_{t=1}^k \sum_{s_{\gamma_{A_t}} < s_{\eta_{A_t}}} \sum_{s_{\gamma_{A_t}} < s_{\eta_{A_t}}} |c_{\gamma_{A_t} \gamma_{A_t}} c_{\eta_{A_t} \eta_{A_t}} - c_{\eta_{A_t} \gamma_{A_t}} c_{\gamma_{A_t} \eta_{A_t}}|^2}{k} \\ &\geq \frac{4 \sum_{t=1}^k \sum_{|\eta_{A_t}|=1, |\eta_{A_t}|=1} |c_{\eta_{A_t} 0_{A_t}} c_{0_{A_t} \eta_{A_t}} - c_{0_{A_t} 0_{A_t}} c_{\eta_{A_t} \eta_{A_t}}|^2}{k},\end{aligned}\quad (15)$$

where $0_{A_t} = (i_{j_1^t}, i_{j_2^t}, \dots, i_{j_{n_t}^t}) = (0, 0, \dots, 0)$, $|\eta_{A_t}|$ and $|\eta_{A_t}|$ represent the numbers of 1 in η_{A_t} , η_{A_t} , respectively.

Next we deal with (15). By using the inequality $n \sum_{i=1}^n |a_i|^2 \geq (\sum_{i=1}^n |a_i|)^2$ (a_i is complex number) and the triangle inequality, we obtain

$$\begin{aligned}\sqrt{\frac{2 \sum_{t=1}^k (1 - \text{Tr}(\rho_{A_t}^2))}{k}} &\geq \frac{2}{\sqrt{k \sum_{t=1}^k n_t(n-n_t)}} \sum_{t=1}^k \sum_{\substack{|\eta_{A_t}|=1 \\ |\eta_{A_t}|=1}} (|c_{\eta_{A_t} 0_{A_t}} c_{0_{A_t} \eta_{A_t}} - c_{0_{A_t} 0_{A_t}} c_{\eta_{A_t} \eta_{A_t}}|) \\ &\geq \frac{2}{\sqrt{k \sum_{t=1}^k n_t(n-n_t)}} \sum_{t=1}^k \sum_{\substack{|\eta_{A_t}|=1 \\ |\eta_{A_t}|=1}} (|c_{\eta_{A_t} 0_{A_t}} c_{0_{A_t} \eta_{A_t}}| - |c_{0_{A_t} 0_{A_t}} c_{\eta_{A_t} \eta_{A_t}}|) \\ &\geq \frac{\sqrt{k}}{\sqrt{\sum_{t=1}^k n_t(n-n_t)}} (2 \sum_{\substack{s_{i_1 \dots i_n} < s_{l_1 \dots l_n} \\ |\{i_1, \dots, i_n\}|=1 \\ |\{l_1, \dots, l_n\}|=1}} |c_{i_1 \dots i_n} c_{l_1 \dots l_n}| \\ &\quad - 2 \sum_{|\{i_1, \dots, i_n\}|=2} |c_{0 \dots 0} c_{i_1 \dots i_n}| - (n-k) \sum_{|\{i_1, \dots, i_n\}|=1} |c_{i_1 \dots i_n}|^2),\end{aligned}\quad (16)$$

from which it follows

$$C_{k-\text{ME}}(|\psi\rangle) = \min_A \sqrt{\frac{2 \sum_{t=1}^k (1 - \text{Tr}(\rho_{A_t}^2))}{k}} \geq H_k Q_k, \quad (17)$$

where

$$H_k = \min_A \frac{\sqrt{k}}{\sqrt{\sum_{t=1}^k n_t(n-n_t)}}, \quad (18)$$

and

$$Q_k = 2 \sum_{\substack{s_{i_1 \dots i_n} < s_{l_1 \dots l_n} \\ |\{i_1, \dots, i_n\}|=1 \\ |\{l_1, \dots, l_n\}|=1}} |c_{i_1 \dots i_n} c_{l_1 \dots l_n}| - 2 \sum_{|\{i_1, \dots, i_n\}|=2} |c_{0 \dots 0} c_{i_1 \dots i_n}| - (n-k) \sum_{|\{i_1, \dots, i_n\}|=1} |c_{i_1 \dots i_n}|^2. \quad (19)$$

Here $|\{i_1, \dots, i_n\}|$ denote the number of $i_l = 1$ in $\{i_1, \dots, i_n\}$.

Now suppose that $\rho = \sum_m p_m \rho^m = \sum_m p_m |\psi_m\rangle \langle \psi_m|$ is an n -partite mixed state where $|\psi_m\rangle = \sum_{i_1, \dots, i_n} c_{i_1 \dots i_n}^m |i_1 \dots i_n\rangle$.

Using (4) and (17), we see

$$C_{k-\text{ME}}(\rho) = \inf_{\{p_m, |\psi_m\rangle\}} \sum_m p_m C_{k-\text{ME}}(|\psi_m\rangle) \geq H_k \inf_{\{p_m, |\psi_m\rangle\}} \sum_m p_m Q_k^m \quad (20)$$

Let $\phi(0) = |00 \dots 0\rangle$ and $0' = 1$, we have

$$I_k(\rho, \phi(0)) = 2 \sum_{i < j} |\rho_{2^{n-i}+1, 2^{n-j}+1}| - 2 \sum_{i < j} \sqrt{\rho_{1,1} \rho_{2^{n-i}+2^{n-j}+1, 2^{n-i}+2^{n-j}+1}} - (n-k) \sum_i \rho_{2^{n-i}+1, 2^{n-i}+1} \quad (21)$$

Considering the three terms of (21), we get

$$\begin{aligned} 2 \sum_{i < j} |\rho_{2^{n-i}+1, 2^{n-j}+1}| &\leq \sum_m p_m (2 \sum_{i < j} |\rho_{2^{n-i}+1, 2^{n-j}+1}^m|) \\ &= \sum_m p_m (2 \sum_{\substack{s_{i_1 \dots i_n} < s_{l_1 \dots l_n} \\ |\{i_1, \dots, i_n\}|=1 \\ |\{l_1, \dots, l_n\}|=1}} |c_{i_1 \dots i_n}^m c_{l_1 \dots l_n}^m|), \end{aligned} \quad (22)$$

$$\begin{aligned} 2 \sum_{i < j} \sqrt{\rho_{1,1} \rho_{2^{n-i}+2^{n-j}+1, 2^{n-i}+2^{n-j}+1}} &= 2 \sum_{i < j} \sqrt{(\sum_m p_m \rho_{1,1}^m)(\sum_m p_m \rho_{2^{n-i}+2^{n-j}+1, 2^{n-i}+2^{n-j}+1}^m)} \\ &\geq \sum_m p_m (2 \sum_{|\{i_1, \dots, i_n\}|=2} |c_{0 \dots 0} c_{i_1 \dots i_n}^m|), \end{aligned} \quad (23)$$

$$(n-k) \sum_i \rho_{2^{n-i}+1, 2^{n-i}+1} = \sum_m p_m (n-k) \sum_{|\{i_1, \dots, i_n\}|=1} |c_{i_1 \dots i_n}^m|^2. \quad (24)$$

Combing (22), (23) and (24), we obtain

$$\begin{aligned} I_k(\rho, \phi(0)) &\leq \sum_m p_m (2 \sum_{\substack{s_{i_1 \dots i_n} < s_{l_1 \dots l_n} \\ |\{i_1, \dots, i_n\}|=1 \\ |\{l_1, \dots, l_n\}|=1}} |c_{i_1 \dots i_n}^m c_{l_1 \dots l_n}^m| - 2 \sum_{|\{i_1, \dots, i_n\}|=2} |c_{0 \dots 0} c_{i_1 \dots i_n}^m| \\ &\quad - (n-k) \sum_{|\{i_1, \dots, i_n\}|=1} |c_{i_1 \dots i_n}^m|^2) \\ &= \sum_m p_m Q_k^m \leq \inf_{\{p_m, |\psi_m\rangle\}} \sum_m p_m Q_k^m. \end{aligned} \quad (25)$$

Observing the relation between (20) and (25), we obtain

$$C_{k-\text{ME}}(\rho) \geq H_k I_k(\rho, \phi(0)). \quad (26)$$

Moreover, for any fully separable state $|\phi(x)\rangle = \otimes_{i=1}^n |x_i\rangle = |x_1 x_2 \dots x_n\rangle$, there exists local unitary transformation $U = U_1 \otimes U_2 \otimes \dots \otimes U_n$ such that $U|\phi(0)\rangle = |\phi(x)\rangle$, thus $H_k I(\rho, \phi(x))$ is also a lower bound because of the invariance of $C_{k-\text{ME}}(\rho)$ under local unitary transformations. Therefore we have

$$C_{k-\text{ME}}(\rho) \geq \max_{\{|\phi(x)\rangle\}} H_k I_k(\rho, |\phi(x)\rangle) \geq H_k I_k(\rho, |\phi(x)\rangle), \quad (27)$$

as desired.

2. Bound 2

By (14), we get

$$\begin{aligned} \frac{2 \sum_{t=1}^k (1 - \text{Tr}(\rho_{A_t}^2))}{k} &= \frac{4 \sum_{t=1}^k \sum_{s \gamma_{A_t} < s \gamma_{\bar{A}_t}} \sum_{\bar{s} \gamma_{\bar{A}_t} < \bar{s} \gamma_{A_t}} |c_{\gamma_{A_t} \gamma_{\bar{A}_t}} c_{\eta_{A_t} \eta_{\bar{A}_t}} - c_{\eta_{A_t} \gamma_{\bar{A}_t}} c_{\gamma_{A_t} \eta_{\bar{A}_t}}|^2}{k} \\ &= \frac{4 \sum_{t=1}^k \left(\sum_{\substack{|\eta_{A_t}|=1 \\ |\eta_{\bar{A}_t}|=1}} |c_{\eta_{A_t} 0_{\bar{A}_t}} c_{0_{A_t} \eta_{\bar{A}_t}} - c_{0_{A_t} 0_{\bar{A}_t}} c_{\eta_{A_t} \eta_{\bar{A}_t}}|^2 + \sum_{\substack{|\eta_{A_t}|=n_t-1 \\ |\eta_{\bar{A}_t}|=n-n_t-1}} |c_{\eta_{A_t} 1_{\bar{A}_t}} c_{1_{A_t} \eta_{\bar{A}_t}} - c_{1_{A_t} 1_{\bar{A}_t}} c_{\eta_{A_t} \eta_{\bar{A}_t}}|^2 \right)}{k}. \end{aligned} \quad (28)$$

As in the proof of bound 1, there is

$$\begin{aligned} \sqrt{\frac{2 \sum_{t=1}^k (1 - \text{Tr}(\rho_{A_t}^2))}{k}} &\geq \frac{2}{\sqrt{2k \sum_{t=1}^k n_t(n-n_t)}} \sum_{t=1}^k \left(\sum_{\substack{|\eta_{A_t}|=1 \\ |\eta_{\bar{A}_t}|=1}} |c_{\eta_{A_t} 0_{\bar{A}_t}} c_{0_{A_t} \eta_{\bar{A}_t}} - c_{0_{A_t} 0_{\bar{A}_t}} c_{\eta_{A_t} \eta_{\bar{A}_t}}| \right. \\ &\quad \left. + \sum_{\substack{|\eta_{A_t}|=n_t-1 \\ |\eta_{\bar{A}_t}|=n-n_t-1}} |c_{\eta_{A_t} 1_{\bar{A}_t}} c_{1_{A_t} \eta_{\bar{A}_t}} - c_{1_{A_t} 1_{\bar{A}_t}} c_{\eta_{A_t} \eta_{\bar{A}_t}}| \right) \\ &\geq \frac{2}{\sqrt{2k \sum_{t=1}^k n_t(n-n_t)}} \sum_{t=1}^k \left[\sum_{\substack{|\eta_{A_t}|=1 \\ |\eta_{\bar{A}_t}|=1}} (|c_{\eta_{A_t} 0_{\bar{A}_t}} c_{0_{A_t} \eta_{\bar{A}_t}}| - |c_{0_{A_t} 0_{\bar{A}_t}} c_{\eta_{A_t} \eta_{\bar{A}_t}}|) \right. \\ &\quad \left. + \sum_{\substack{|\eta_{A_t}|=n_t-1 \\ |\eta_{\bar{A}_t}|=n-n_t-1}} (|c_{\eta_{A_t} 1_{\bar{A}_t}} c_{1_{A_t} \eta_{\bar{A}_t}}| - |c_{1_{A_t} 1_{\bar{A}_t}} c_{\eta_{A_t} \eta_{\bar{A}_t}}|) \right] \\ &\geq \frac{\sqrt{k}}{\sqrt{2 \sum_{t=1}^k n_t(n-n_t)}} \left(2 \sum_{\substack{s_{i_1 \dots i_n} < s_{l_1 \dots l_n} \\ |\{i_1, \dots, i_n\}|=1 \\ |\{l_1, \dots, l_n\}|=1}} |c_{i_1 \dots i_n} c_{l_1 \dots l_n}| - 2 \sum_{|\{i_1, \dots, i_n\}|=2} |c_{0 \dots 0} c_{i_1 \dots i_n}| \right. \\ &\quad \left. - (n-k) \sum_{|\{i_1, \dots, i_n\}|=1} |c_{i_1 \dots i_n}|^2 \right. \\ &\quad \left. + 2 \sum_{\substack{s_{i_1 \dots i_n} < s_{l_1 \dots l_n} \\ |\{i_1, \dots, i_n\}|=n-1 \\ |\{l_1, \dots, l_n\}|=n-1}} |c_{i_1 \dots i_n} c_{l_1 \dots l_n}| - 2 \sum_{|\{i_1, \dots, i_n\}|=n-2} |c_{1 \dots 1} c_{i_1 \dots i_n}| - (n-k) \sum_{|\{i_1, \dots, i_n\}|=n-1} |c_{i_1 \dots i_n}|^2 \right) \end{aligned} \quad (29)$$

So, we get

$$\begin{aligned} C_{k-\text{ME}}(|\psi\rangle) &= \min \sqrt{\frac{2 \sum_{t=1}^k (1 - \text{Tr}(\rho_{A_t}^2))}{k}} \\ &\geq \bar{H}_k^\gamma(Q_k + \bar{Q}_k), \end{aligned} \quad (30)$$

where

$$\bar{H}_k = \min_{\gamma} \frac{\sqrt{k}}{\sqrt{2 \sum_{t=1}^k n_t(n-n_t)}} = \frac{H_k}{\sqrt{2}}, \quad (31)$$

$$Q_k = 2 \sum_{\substack{s_{i_1 \dots i_n} < s_{l_1 \dots l_n} \\ |\{i_1, \dots, i_n\}|=1 \\ |\{l_1, \dots, l_n\}|=1}} |c_{i_1 \dots i_n} c_{l_1 \dots l_n}| - 2 \sum_{|\{i_1, \dots, i_n\}|=2} |c_{0 \dots 0} c_{i_1 \dots i_n}| - (n-k) \sum_{|\{i_1, \dots, i_n\}|=1} |c_{i_1 \dots i_n}|^2. \quad (32)$$

$$\bar{Q}_k = 2 \sum_{\substack{s_{i_1 \dots i_n} < s_{l_1 \dots l_n} \\ |\{i_1, \dots, i_n\}|=n-1 \\ |\{l_1, \dots, l_n\}|=n-1}} |c_{i_1 \dots i_n} c_{l_1 \dots l_n}| - 2 \sum_{|\{i_1, \dots, i_n\}|=n-2} |c_{1 \dots 1} c_{i_1 \dots i_n}| - (n-k) \sum_{|\{i_1, \dots, i_n\}|=n-1} |c_{i_1 \dots i_n}|^2. \quad (33)$$

Now suppose that $\rho = \sum_m p_m \rho^m = \sum_m p_m |\psi_m\rangle \langle \psi_m|$ is an n -partite mixed state where $|\psi_m\rangle = \sum_{i_1, \dots, i_n} c_{i_1 \dots i_n}^m |i_1 \dots i_n\rangle$.

Using (4) and (30), we see

$$C_{k-\text{ME}}(\rho) = \inf_{\{p_m, |\psi_m\rangle\}} \sum_m p_m C_{k-\text{ME}}(|\psi_m\rangle) \geq \bar{H}_k \inf_{\{p_m, |\psi_m\rangle\}} \sum_m p_m (Q_k^m + \bar{Q}_k^m). \quad (34)$$

Let $|\phi(1)\rangle = |11\cdots 1\rangle$ and $1' = 0$, then there is

$$I_k(\rho, \phi(1)) = 2 \sum_{i < j} |\rho_{2^n-2^{n-i}, 2^n-2^{n-j}}| - 2 \sum_{i < j} \sqrt{\rho_{2^n, 2^n} \rho_{2^n-2^{n-i}-2^{n-j}, 2^n-2^{n-i}-2^{n-j}}} - (n-k) \sum_i \rho_{2^n-2^{n-i}, 2^n-2^{n-i}} \quad (35)$$

For the first term of (35),

$$2 \sum_{i < j} |\rho_{2^n-2^{n-i}, 2^n-2^{n-j}}| \leq \sum_m p_m (2 \sum_{\substack{s_{i_1 \dots i_n} < s_{l_1 \dots l_n} \\ |\{i_1, \dots, i_n\}| = n-1 \\ |\{l_1, \dots, l_n\}| = n-1}} |c_{i_1 \dots i_n}^m c_{l_1 \dots l_n}^m|). \quad (36)$$

For the second term,

$$\begin{aligned} 2 \sum_{i < j} \sqrt{\rho_{2^n, 2^n} \rho_{2^n-2^{n-i}-2^{n-j}, 2^n-2^{n-i}-2^{n-j}}} &= 2 \sum_{i < j} \sqrt{(\sum_m p_m \rho_{2^n, 2^n}^m) (\sum_m p_m \rho_{2^n-2^{n-i}-2^{n-j}, 2^n-2^{n-i}-2^{n-j}}^m)} \\ &\geq \sum_m p_m (2 \sum_{|\{i_1, \dots, i_n\}| = n-2} |c_{1 \dots 1}^m c_{i_1 \dots i_n}^m|). \end{aligned} \quad (37)$$

For the third term,

$$(n-k) \sum_i \rho_{2^n-2^{n-i}, 2^n-2^{n-i}} = \sum_m p_m [(n-k) \sum_{|\{i_1, \dots, i_n\}| = n-1} |c_{i_1 \dots i_n}|^2]. \quad (38)$$

Combing (36), (37) and (38), we obtain

$$I_k(\rho, \phi(1)) \leq \sum_m p_m \bar{Q}_k^m \leq \inf_{\{p_m, |\psi_m\rangle\}} \sum_m p_m \bar{Q}_k^m \quad (39)$$

Observing the relation between (34), (25) and (39), we obtain

$$C_{k-\text{ME}}(\rho) \geq \bar{H}_k(I_k(\rho, |0\rangle, |1\rangle) + I_k(\rho, |1\rangle, |0\rangle)) \quad (40)$$

Note that for any fully separable state $|\phi(y)\rangle = \otimes_{i=1}^n |y_i\rangle$, there is local unitary transformation $V = V_1 \otimes V_2 \otimes \cdots \otimes V_n$ satisfying $V|\phi(0)\rangle = |\phi(y)\rangle$. Thus $\bar{H}_k(I(\rho, \phi(x)) + I(\rho, \phi(y)))$ is also a lower bound because of the invariance of $C_k(\rho)$ under local unitary transformations, so we have

$$C_{k-\text{ME}}(\rho) \geq \max_{\{\phi(x), \phi(y)\}} \bar{H}_k((I(\rho, \phi(x)) + I(\rho, \phi(y)))) \geq \bar{H}_k(I(\rho, \phi(x)) + I(\rho, \phi(y))). \quad (41)$$

The proof is complete.

C. Examples

Example 1: Consider the n -qubit state family given by a mixture of the identity matrix, the W state and the anti- W state

$$\rho_n = \frac{1-2a}{2^n} I_{2^n} + a |W_n\rangle\langle W_n| + a |\tilde{W}_n\rangle\langle \tilde{W}_n| \quad (42)$$

where $|W_n\rangle = \frac{1}{\sqrt{n}}(|00\cdots 001\rangle + |00\cdots 010\rangle + \cdots + |10\cdots 000\rangle)$ and $|\tilde{W}_n\rangle = \frac{1}{\sqrt{n}}(|11\cdots 110\rangle + |11\cdots 101\rangle + \cdots + |01\cdots 111\rangle)$. Let $|\phi(0)\rangle = |0\rangle^{\otimes n}$, then $|\phi_i(0)\rangle = |0\cdots 010\cdots 0\rangle$ by applying the bit-flip operation σ_x on the i -th qubit of $|\phi(0)\rangle$. Our Bound 1 is as follows:

$$\begin{aligned} C_{2-\text{ME}}(\rho) &\geq \min_{n_1 \in \{1, 2, \dots, n-1\}} \frac{1}{\sqrt{n_1(n-n_1)}} \left(a - \frac{n(2n-3)(1-2a)}{2^n} \right) \\ &\geq \begin{cases} \frac{2}{n} \left(a - \frac{n(2n-3)(1-2a)}{2^n} \right), & n \text{ is even,} \\ \frac{2}{\sqrt{n^2-1}} \left(a - \frac{n(2n-3)(1-2a)}{2^n} \right), & n \text{ is odd.} \end{cases} \end{aligned} \quad (43)$$

Our Bound 2 gives

$$C_{2-\text{ME}}(\rho) \geq \begin{cases} \frac{2\sqrt{2}}{n} \left(a - \frac{n(2n-3)(1-2a)}{2^n} \right), & n \text{ is even,} \\ \frac{2\sqrt{2}}{\sqrt{n^2-1}} \left(a - \frac{n(2n-3)(1-2a)}{2^n} \right), & n \text{ is odd.} \end{cases} \quad (44)$$

The bound 1 of Ref.[22] and bound of [21] are as follows, respectively,

$$C_{\text{GME}}(\rho) \geq \frac{1}{\sqrt{2}(n-1)} \left(a - \frac{n(2n-3)(1-2a)}{2^n} \right), \quad (45)$$

$$C_{\text{GME}}(\rho) \geq \frac{1}{n-1} \left(a - \frac{n(2n-3)(1-2a)}{2^n} \right). \quad (46)$$

Obviously, both of our lower bounds are better than the bound 1 of Ref.[22] and bound of [21].

Example 2: Let consider the family of five-qubit states

$$\rho = \frac{1-a-b}{32} I_{32} + a|W_5\rangle\langle W_5| + b|\tilde{W}_5\rangle\langle \tilde{W}_5|, \quad (47)$$

the mixture of the identity matrix, the W state and the anti- W state. Here $|W_5\rangle = \frac{1}{\sqrt{5}}(|00001\rangle + |00010\rangle + |00100\rangle + |01000\rangle + |10000\rangle)$ and $|\tilde{W}_5\rangle = \frac{1}{\sqrt{5}}(|11110\rangle + |11101\rangle + |11011\rangle + |10111\rangle + |01111\rangle)$. For this family, the detection quality of our bound 1 is better than the bound 1 of [21].

Selecting $|\phi\rangle = |0\rangle^{\otimes n}$ or $|\phi\rangle = |1\rangle^{\otimes n}$ and $U_i = \sigma_x$, From our Bound 1 Ineq.(27), there are,

$$C_{2-\text{ME}}(\rho) \geq \frac{67b + 35a - 35}{32\sqrt{6}}, \quad (48)$$

$$C_{2-\text{ME}}(\rho) \geq \frac{67a + 35b - 35}{32\sqrt{6}}, \quad (49)$$

By the bound 1 of Ref [22], there are,

$$C_{\text{GME}}(\rho) \geq \frac{67b + 35a - 35}{128}, \quad (50)$$

$$C_{\text{GME}}(\rho) \geq \frac{67a + 35b - 35}{128}. \quad (51)$$

The detection parameter spaces of our obtained bounds and bound 1 in [21] on the GME-concurrence is illustrated in Fig. 2 for the family ρ_5 of five-qubit states. The area detected by our bound 1 is larger than the bound 1 of [21] when the two lower bounds are equal.

V. EXPERIMENTAL IMPLEMENTATION

Our conclusion is experimental accessible by means of local observables, without quantum state tomography which requires an exponentially increasing measurements. In fact, the lower bound of (27) and (41), for any fixed product states $|\phi(x)\rangle$, only needs at most $\frac{5(n^2-n)}{2} + n + 1$ local observables and the specific implementing is shown in [16] and [18].

VI. CONCLUSION

We have defined a measure of multipartite entanglement called k -ME concurrence and studied multipartite entanglement of quantum states in arbitrary dimensional systems. Two lower bounds of k -ME concurrence $C_{k-\text{ME}}(\rho)$ for n -partite mixed quantum states through the inequality (3) from Ref.[18] are given. We provide examples in which the lower bounds perform better than the previously known methods.

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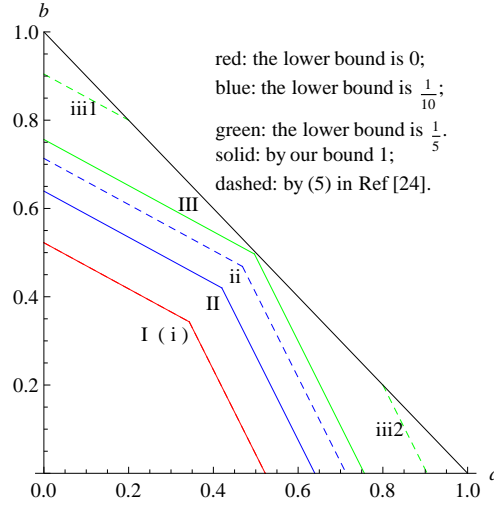


FIG. 2: (Color online). The detection quality of our lower bounds and bound 1 in [21] on the GME-concurrence is shown for the family $\rho_5 = \frac{1-a-b}{32}I_{32} + a|W_5\rangle\langle W_5| + b|\tilde{W}_5\rangle\langle \tilde{W}_5|$ of five-qubit states, where $|W_5\rangle = \frac{1}{\sqrt{5}}(|00001\rangle + |00010\rangle + |00100\rangle + |01000\rangle + |10000\rangle)$ and $|\tilde{W}_5\rangle = \frac{1}{\sqrt{5}}(|11110\rangle + |11101\rangle + |11011\rangle + |10111\rangle + |01111\rangle)$. The area above the line I (red) is the genuine 5-partite entanglement area of ρ_5 detected by our bound 1, our criteria in [16, 18], and the bound 1 of [22]. The states in the areas above solid line II (blue) (line III (green)) are genuine 5-partite entangled detected by our lower bound 1 when the bound is equal to or greater than $\frac{1}{10}$ ($\frac{1}{5}$). The states above the dashed line ii (blue), the dashed line iii1 (green) and the dashed line iii2 (green), are detected by the bound 1 of Ref.[22] when it is equal to or greater than $\frac{1}{10}$, $\frac{1}{5}$ and $\frac{1}{5}$, respectively. Thus, the area detected by our bound 1 is visibly larger than that of [22] when the two bounds are equal.

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